4-25-2018

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Recommended Citation

Jongsma, C. (2018). Abstraction and Infinity (Book Review). *MAA Reviews Online* Retrieved from https://digitalcollections.dordt.edu/faculty_work/892

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Abstract
Reviewed Title: *Abstraction and Infinity* by Paolo Mancosu, Oxford University Press, 2017. 222 pp. ISBN: 9780198746829.

Keywords
book review, Paolo Mancosu, Abstraction and Infinity

Disciplines
Mathematics

Abstraction and Infinity by Paulo Mancosu

Publisher: Oxford University Press

Publication Date: 2017

Number of Pages: 222

Format: Hardcover

Price: 70.00

ISBN: 9780198746829

Category: Monograph

Reviewed by Calvin Jongsma, on 04/25/2018

Paolo Mancosu's *Abstraction and Infinity* expertly straddles the history and philosophy of mathematics. Drawing heavily from several earlier publications (2009 ff.), Mancosu explores definition by abstraction (abstraction principles) and different notions of infinity (cardinality, numerosity) from historical, mathematical, and philosophical perspectives. Readers will find these topics interesting in their own right, but Mancosu's main motivation for studying them is to deepen and advance an ongoing dialogue on the nature and viability of neo-logicism, a contemporary viewpoint on the foundations of arithmetic.

To briefly sketch some context, Frege's logicist program of reducing mathematics to logic was undertaken in several works during the late nineteenth century (logical foundations: 1879; philosophy of arithmetic: 1884; axiomatic foundations of arithmetic: 1893, 1903). A bedrock proposition in Frege's attempt was his fated Basic Law V, which was key to his developing a theory of arithmetic. This is the axiom whose inconsistency Bertrand Russell alerted him to in a letter of 1902. Frege soon despaired of repairing his approach, but Russell and Whitehead took up the quest of reducing mathematics to logic, producing their three-volume *Principia Mathematica* in two editions between 1910 and 1927. In the end, however, they too admitted they had not been able to accomplish what they had hoped because they had found it necessary to assume some axioms whose logical character and truth were questionable.

A version of Frege's logicism resurfaced after Crispin Wright convincingly argued in 1983 that the project could be resuscitated in the context of second-order logic by assuming Hume's Principle (HP), which states that *the number of Fs equals the number of Gs if and only if the Fs are in one-to-one correspondence with the Gs*. This is the criterion Cantor adopted for his investigations into transfinite cardinal numbers and a result Frege had deduced from his Law V in order to prove the Dedekind-Peano Postulates. In the last few decades, debates over neo-logicist proposals have focused on the merits of HP, whether it's the right principle to use as a foundation for arithmetic, what properties it has that distinguish it from alternatives in conflict with it ("bad companions"), and whether it can be considered a logical or analytic truth.

Basic Law V and HP are abstraction principles, that is, they postulate or identify objects whose existence arises from satisfying an equivalence relation — numbers being abstracted from the equinumerosity of concepts' extensions (classes) in the case of HP. Mancosu's research on abstraction principles provides a context pertinent to the debates surrounding neo-logicist developments as well as for revisiting Frege's work and Cantor's treatment of cardinality.

Mancosu notes that abstraction principles operate in three main ways. In number theory, for example, integers that are congruent modulo n lead to integers mod n. These can be conceived of either as residues resulting from division by n (*representative* integers) or as *equivalence classes* of numbers sharing the same remainder. In geometry, lines can be said to have the same direction if they are parallel (or identical). Here the equivalence classes lack a natural representative, so any particular choice would seem artificial. Direction can then either be identified with the *equivalence class* itself or postulated as

some *new entity* associated with the class. Over the centuries, mathematicians have taken all three tacks in connection with abstraction principles — choosing class representatives, using equivalence classes, or postulating new entities. Dedekind, for example, believed (1872) that the human mind creates irrationals as wholly new numbers via his cut principle (a real number is uniquely determined by an order-preserving partition of the rationals into two infinite intervals). Russell insisted instead on identifying a real number with its cut, retorting (1919) that "the method of 'postulating' what we want has many advantages; they are the same as the advantages of theft over honest toil." The notion of cardinal number illustrates all three options. Cantor thought of cardinal numbers as abstract entities determined by equinumerosity, created by abstracting both from the nature of a set's elements and from their order. The obvious alternative to identify cardinal numbers with their associated equivalence classes succumbed to Russell's Paradox (the class of all equinumerous classes is contradictory), so in the end cardinal numbers were identified (following von Neuman in 1923) with the least ordinal number of that size, a representative object in the class.
Frege seems to have been the first mathematician to explicitly emphasize abstraction principles leading to equivalence classes, though he was not the first to make use of them. Mancosu devotes his first chapter to tracing earlier uses of definition by abstraction, focusing mainly on nineteenth-century occurrences in number theory, algebra, geometry, physics, and set theory.

The second chapter analyzes Frege's own use of the principle and details various geometry textbooks' treatment of direction that Frege may have been aware of as he developed his own ideas. The chapter concludes by surveying the views of Peano and his colleagues Burali-Forti and Padoa, who were instrumental in analyzing and promoting definition by abstraction, and by looking at Russell's conscious use of such definitions in his *Principles of Mathematics*(1903).
Chapter three samples some standpoints throughout history regarding the paradoxes of infinity as exemplified by Galileo's well-known paradox: the collection of natural numbers is both larger than that of their squares according to the Part-Whole Principle (Euclid's Axiom 5: the whole is greater than its part), while simultaneously being equinumerous to it. Starting with the Greek rejection of the possibility of different sized infinities (Proclus, Philoponus), Mancosu continues by looking at a few medieval Arabic and Latin thinkers who took a variety of positions on comparing infinities. He then discusses the viewpoints of Galileo, Leibniz, and the lesser known Spanish natural philosopher Maignan before turning to the ideas of Bolzano, Dedekind, and Cantor in the nineteenth century.

The third chapter concludes with an exposition of some technical developments around the turn of the twenty-first century that extend the Part-Whole Principle beyond its use for finite sets into a theory of numerosities that distinguish the size of certain infinite sets having the same cardinality. In such theories, the set of squares has numerosity strictly less than the set of natural numbers, and this holds for any two sets standing in a strict inclusion relation. There are some peculiarities attending such a theory at this stage (it only applies to infinite sets of limited sizes, and the numerosity relations between sets is dependent upon how the sets are counted, or, equivalently, upon the ultrafilter used), but its existence indicates that Cantor's approach to infinity via the equinumerosity abstraction principle is not the only viable or inevitable option, as Gödel once claimed (1947).

Mancosu also notes (in the book's introduction) that other heterodox views on infinity existed around the turn of the twentieth century, when various Italian mathematicians and others explored non-Archimedean systems of numbers. These systems made use of infinitesimals in various parts of mathematics, predating Abraham Robinson's model-theoretic approach to non-standard analysis in the mid-1960s.

The final chapter of the book applies its earlier findings along with some further historical analysis of nineteenth-century thinking about the size of infinite sets (Bolzano, Cantor, Dedekind, Frege, Schröder, Peano) to current philosophical debates about the use of Hume's Principle in neo-logicist circles. Mancosu constructs a countably infinite number of alternative abstraction principles to HP that can be used as a deductive basis for second-order arithmetic, principles sharing all the good properties attributed to HP ("good companions"). Mancosu uses these to highlight questions about the analyticity of HP as well as HP's pride of place among the various alternatives that can be used as a basis for arithmetic. He concludes by pointing out that neo-logicists might take different stands in addressing these matters.

Mancosu's analysis clarifies some issues associated with recent developments and moves the debate about neo-logicism forward.

Mancosu's work is well written and has been well received by those involved with neo-logicism. But it also offers readers not engaged in such debates an opportunity to reflect on the fact that of two intuitively true principles for comparing finite collections (one-to-one correspondence and part-whole inclusion), the historic choice of equinumerosity by Cantor for investigating infinite collections was and is not as cut-and-dried as some believe. And, of course, the book provides a valuable scholarly account of the history of definition by abstraction, the use of equivalence relations, and notions of infinity.

Acknowledgements: I'd like to thank Paolo Mancosu for offering some helpful feedback to an earlier draft of this review.